Spin Hall Effect For Anyons

S. Dhar*
Physics Department, Bankura Sammilani College, West Bengal, India

B. Basu† and Subir Ghosh‡
Physics and Applied Mathematics Unit
Indian Statistical Institute
Kolkata-700108, India

We explain the intrinsic spin Hall effect from generic anyon dynamics in the presence of external electromagnetic field. The free anyon is represented as a spinning particle with an underlying non-commutative configuration space. The Berry curvature plays a major role in the analysis.

PACS numbers: 73.43.-f, 72.25.Dc., 03.65.Vf

I. INTRODUCTION

In the nascent field of spintronics [1] understanding the dynamics of spin current is extremely important. The prediction [2] and observation [3] of an intrinsic Spin Hall Effect has evoked a lot of interest since the dissipationless Spin Hall current can be an efficient means of injecting spin current in (Ga-As) semiconductors. Furthermore the Spin Hall conductivity can be quantized [4] and the resulting quantum spin Hall liquid will have exotic features such as *fractional* statistics indicating the presence of *anyons* [5] as quasi-particles. In fact signatures of anyons in Ga-As heterostructures have been reported [6]. An *anomalous velocity* component of electrons, proposed long ago in [7], is responsible for Spin Hall [2] and Anomalous Hall Effect [8]. The *semi-classical* analysis of Bloch electrons in solid in the presence of external electromagnetic fields, pioneered by Chang and Niu [9, 10], has shown that this anomalous velocity is induced by the intrinsic Berry curvature [11] in Bloch bands [12]. This discussion brings out the perspective of the work reported in this Letter.

The Berry phase emerges during the evolution of a particle with a spin by introducing a spin gauge field. Such a particle with a spin can be described by a vector (multicomponent) wavefunction [10]. Anyons possess arbitrary spin [5] and, conforming to the above idea, this makes them a prime candidate in Berry phase study. Indeed, an analogue of the Dirac equation, to describe a free relativistic anyon, has been formulated [13] that requires an infinite component wavefunction. However, instead of exploiting the multicomponent anyon wavefunction of [13], we will employ the *spinning particle* model [14, 15, 16]. In this model the anyon wavefunction is a *single component scalar* and the arbitrary spin is induced by the underlying NC configuration spacetime of anyon. The NC parameter appears as the anyon spin. The *non-Abelian* nature of the $U(1)$ gauge theory in NC spacetime has already been noticed [17]. We find its echo in the condensed matter scenario where the Berry gauge field assumes a non-Abelian character.

We study dynamics of anyons in the presence of external electromagnetic field [14, 15, 16]. Their behavior is quite different from that of a point charge due to the inherent spin-orbit coupling in the former. We interpret this system as an effective model of semi-classical dynamics [9] of anyon excitations in the Ga-As alloy.

We start by representing the general framework of anyon phase space and identify the Berry phase. Then we consider a particular anyon model mentioned above [14, 15, 16] and obtain the explicit forms of the Berry potentials. We show that the Berry phase plays an important role to obtain the anomalous velocity and also spin Hall conductivity.

*Electronic address: sarmi˙30@rediffmail.com
†Electronic address: banasri@isical.ac.in
‡Electronic address: subir˙ghosh2@rediffmail.com

II. GENERAL FRAMEWORK

We treat (free relativistic) anyons as $2+1$-dimensional spinning particles and consider them in an external electromagnetic field (E_1, E_2, B). In this configuration the noncommutative geometry play the key role of the anyon configuration space (or more generally phase space) [14, 15, 16] and very interestingly Berry curvature emerges here through NC geometry. The interesting connection between NC spacetime and Berry curvature effect was demonstrated recently in [21], in the context of momentum space singularity in anomalous Hall effect [8].

The anyon phase space variables (r_μ, p_μ) are the covariant physical degrees of freedom. But, as mentioned above, they obey an NC algebra that reduces to the canonical phase space Poisson brackets for $s = 0$. Because of this (r_μ, p_μ) can not be used directly in the Einstein-Brillouin-Keller quantization scheme [9] or in the identification of the Berry potentials [10]. However, one can "solve" the NC algebra in terms of a *canonical* (Darboux) set of variables $[R_\mu, P_\nu] = -ig_{\mu\nu}$, $[R_\mu, R_\nu] = [P_\mu, P_\nu] = 0$. The set (R_μ, P_μ) can be used in the Einstein-Brillouin-Keller framework. From the mapping between (r_μ, p_μ) and (R_μ, P_μ) the all important Berry potentials A^p_μ, A^r_μ can be simply read off [10]. However, to compare with experiments, we have to finally re-express the results in terms of the physical degrees of freedom (r_μ, p_μ). Explicitly, the Darboux transformation, Berry potential and curvature components $\Omega^{rp}_{\mu\nu}$ etc. are defined as in [10],

$$R_\mu \equiv r_\mu - A^p_\mu\ ,\ P_\mu \equiv p_\mu + A^r_\mu;\quad \Omega^{rp}_{\mu\nu} = [p_\mu, A^p_\nu] - [A^r_\mu, r_\nu] + [A^r_\mu, A^p_\nu]. \tag{1}$$

The other curvature components follow in an obvious way. Notice that the NC phase space requires a more general definition of the curvature $\Omega^{rp}_{\mu\nu}$ than the one prescribed in [10], apart from the non-abelian $[A^r, A^p]$ commutator term. The Hamiltonian equations of motion are,

$$\dot{p}_\mu = -i[p_\mu, G]\ ;\quad \dot{r}_\mu = -i[r_\mu, G], \tag{2}$$

where G is generator of (relativistic) time evolution.

The Einstein-Brillouin-Keller quantization condition is straightforward in terms of canonical (R_μ, P_μ) coordinates:

$$\oint \vec{P}.d\vec{R} = j + \frac{\nu}{4}, \tag{3}$$

where j is an integer and ν the Maslov index. In terms of (r_μ, p_μ), the physical variables, (3) read,

$$\oint \vec{P}.d\vec{R} = \oint (\vec{p} + \vec{A}^r).d(\vec{r} - \vec{A}^p) = \oint \vec{p}.d\vec{r} + \oint (\vec{A}^r.d\vec{r} + \vec{A}^p.d\vec{p}), \tag{4}$$

where geometric part of $\varphi = \oint(\vec{A}^r.d\vec{r} + \vec{A}^p.d\vec{p})$ is generally termed as the Berry phase.

III. ANYON MODEL

After discussing the general setup we now turn to the particular case at hand where the anyon wave-function is a single component scalar and the arbitrary spin is induced by the NC parameter. We will restrict ourselves to the lowest non-trivial order in the electromagnetic coupling e and consider a generalized anyon model [16] with arbitrary gyromagnetic ratio g. In these anyon models [14, 15, 16] the spin tensor $S_{\mu\nu}$ is not independent, $S_{\mu\nu} = s\epsilon_{\mu\nu\lambda}p_\lambda/\sqrt{p^2}$ where s and p_μ are the arbitrary spin parameter and momentum respectively. Hence spin operators can always be replaced by momentum operators and spin effects, *e.g.* in equations of motion, derived using the NC phase space algebra), are identified through the parameter s.

The anyon dynamics is governed by the following generator G [16] and $O(e)$ NC phase space brackets [14, 15, 16]:

$$G = -\frac{1}{2m}(p^2 - m^2 + \frac{ge}{2}S_{\mu\nu}F^{\mu\nu}) = -\frac{1}{2m}[p^2 - \{m^2 - \frac{ges}{p}(\vec{p}\times\vec{E} + Bp_0)\}], \tag{5}$$

$$[r_\mu, r_\nu] = if_{\mu\nu} - ie(fFf)_{\mu\nu} \ ; \ [p_\mu, r_\nu] = ig_{\mu\nu} - i(Ff)_{\mu\nu} \ ; \ [p_\mu, p_\nu] = ieF_{\mu\nu}, \tag{6}$$

where $f_{\mu\nu} = s\epsilon_{\mu\nu\sigma}p^\sigma/(p^2)^{\frac{3}{2}}$, the metric is $g_{00} = -g_{ii} = 1$ and $F_{i0} = E_i, F_{ij} = \epsilon_{ij}B$. With this algebra (6), the Lorentz generator J_μ that transforms r_μ, p_μ correctly, contains a spin-part [14, 15],

$$J_\mu = \epsilon_{\mu\nu\lambda}r^\nu p^\lambda + sp_\mu/p, \tag{7}$$

and is structurally very similar to the angular momentum defined by Murakami et.al. in [8].

The canonical coordinates (R_μ, P_μ) to $O(e)$ are computed in terms of the physical (r_μ, p_μ) variables, by exploiting the relations derived in [14] and we obtain,

$$P_\mu = p_\mu + \frac{e}{2}F_{\mu\nu}(r^\nu + s\alpha^\nu[p]) \equiv p_\mu + A^r_\mu,$$

$$R_\mu = r_\mu - s\alpha_\mu[\tilde{p}] - \frac{es}{2}F_{\rho\nu}\{(r^\rho - s\alpha^\rho[p])\frac{\partial\alpha_\mu}{\partial p_\nu} + s\alpha^\rho\frac{\partial\alpha^\nu}{\partial p^\mu}\} \equiv r_\mu - A^p_\mu \tag{8}$$

where

$$\alpha_\mu[p] = (\epsilon_{\mu\nu\rho}p^\nu\eta^\rho)/\lambda, \quad \lambda = p^2 + \sqrt{p^2}p_0, \quad \eta_\mu = \{1, 0, 0\},$$

$$\tilde{p}_\mu = p_\mu + \frac{e}{2}F_{\mu\nu}(r^\nu + s\alpha^\nu[p])$$

and the Berry potentials are introduced. Explicit forms of the potentials, to $O(e)$, are,

$$A^r_0 = \frac{e}{2}(\vec{E}.\vec{r} - \frac{s}{\lambda}\epsilon_{ij}E_ip_j) \ , \ A^r_i = \frac{e}{2}(r_0E_i - B(\epsilon_{ij}r_j + \frac{s}{\lambda}p_i)), \tag{9}$$

$$A^p_0 = 0 \ , \ A^p_i = -\frac{s}{\lambda}(1 + \frac{esB}{2\lambda})\epsilon_{ij}p_j. \tag{10}$$

Thus the Berry curvatures induce the NC brackets (6):

$$[r_i, r_j] = i\frac{sp_0}{p^3}[\epsilon_{ij} + \frac{es}{p^3}(p_iE_j - p_jE_i + p_0B\epsilon_{ij})] \equiv i\Omega^{pp}_{ij} \ ; \ [p_i, p_j] = ieB\epsilon_{ij} \equiv -i\Omega^{rr}_{ij},$$

$$[p_i, r_j] = -i\delta_{ij} + i\frac{es}{p^3}(\epsilon_{jk}E_ip_k - Bp_0\delta_{ij}) \equiv i\delta_{ij} - i\Omega^{rp}_{ij}, \quad etc. \tag{11}$$

It should be noted that external $F_{\mu\nu}$ are also included in our definition of the curvature $\Omega_{\mu\nu}$. $e = 0$ and es-terms yield the purely intrinsic part and spin-dependent part respectively in the curvature. Clearly, even to $O(e)$, the commutator terms $[A_\mu, A_\nu]$ are non-zero which shows the non-Abelian nature of the curvature like $U(1)$ gauge theory in NC space time [17]. In our formalism, this is due to the underlying NC geometry (6,11).

Next we derive the equations of motion (see also [16]) by using (2,5,11):

$$\dot{r}_i = \frac{p_i}{m} + (1 - \frac{g}{2})\frac{es}{mp}[\epsilon_{ij}E_j + (p_0B + \vec{p}\times\vec{E})\frac{p_i}{p^2})], \quad \dot{p}_i = \frac{e}{m}(-p_0E_i + B\epsilon_{ij}p_j). \tag{12}$$

Notice that in the *anomalous* (or non-canonical) part of the velocity equation for $\dot{r}_i$, the g-term comes from the Hamiltonian and the rest is contributed by the coordinate-momentum mixed curvature Ω^{rp} whereas the canonical Lorentz force equation for $\dot{p}_i$ is induced by Ω^{rr}.

IV. PHYSICAL SIGNIFICANCE

In the Berry phase contribution, the intrinsic term (in the non-relativistic limit),

$$\varphi\mid_{e=0}=\oint\vec{A}^{p}(e=0)d\vec{p}=\frac{s}{m^{2}}\times area\ of\ p-orbit \tag{13}$$

will modify the energy spectra [9] and density of states [12] of the excitation. Indeed, there are further spin-orbit (es) contributions in φ as well, that can be generated from (9,10). The equations of motion (12) explicitly shows the effect of spin-orbit coupling. The "exotic" particle model studied in [16, 18], which is a generalization of the anyon with an anomalous gyromagnetic ratio, also has a similar spin orbit interaction [18].

From the equations of motion, in the non-relativistic limit, we get,

$$\dot{r}_{i}=[1+(1-\frac{g}{2})\frac{es}{m^{3}}\vec{p}\times\vec{E}]\frac{p_{i}}{m}+(1-\frac{g}{2})\frac{s}{m^{2}}\epsilon_{ij}\dot{p}_{j} \tag{14}$$

where the last term is the anomalous velocity component. So anyon dynamics in the presence of external electromagnetic field naturally yields the anomalous velocity and the Berry curvature in *mixed* position-momentum space, (first highlighted in [19]), plays an important role in determining the *anomalous* velocity.

On the other hand, coming to the Hall effect considerations, we rewrite the velocity equation in (14) in the form,

$$\dot{r}_{i}=[1+(1-\frac{g}{2})\frac{esB}{m^{2}}]\frac{p_{i}}{m}+(1-\frac{g}{2})\frac{es}{m^{2}}\epsilon_{ij}E_{j}\equiv\frac{p_{i}}{m^{*}}+(1-\frac{g}{2})\frac{es}{m^{2}}\epsilon_{ij}E_{j}\equiv, \tag{15}$$

where the $O(1/m^{4})$ term is dropped. The effective mass is now m^{*} and the last term signifies Hall motion since it induces a velocity, transverse to the Electric field. For $\vec{E}=(E_{x},0)$ we find

$$\dot{x}=\frac{p_{x}}{m^{*}},\quad \dot{y}=\frac{p_{y}}{m^{*}}-(1-\frac{g}{2})\frac{es}{m^{2}}E_{x}. \tag{16}$$

Clearly the Hall conductivity is given by [20]

$$\sigma_{xy}=j_{y}/E_{x}=(e\dot{y})/E_{x}=(1-\frac{g}{2})\frac{se^{2}}{m^{2}}\equiv(e^{*})^{2}\nu. \tag{17}$$

Here, e^{*} can be considered as the effective (or fractional) charge. The parameter ν is the intrinsic Berry phase (13) and as expected it also appears in the intrinsic NC coordinate algebra: $[x_{1},x_{2}]=is/m^{2}\sim i\nu$ and following [20] it can be identified as the "magnetic length". Hence, as we set out to demonstrate, *the Hall conductivity σ_{xy} is given by the Berry curvature.*

Similarly we can estimate the spin Hall conductivity to be,

$$\sigma^{s}_{xy}=J_{y}/E_{x}=(s\frac{|\vec{p}|}{m}\dot{y})/E_{x}=\sqrt{1-\frac{g}{2}}e^{*}s\frac{|\vec{p}|}{m}\nu. \tag{18}$$

Typically, $|\vec{p}|$ appears as the Fermi momentum (see Murakami et.al. in [2]).

From the experimental observations of fractional charge and Hall conductivity, it is possible to obtain values of s and g which are anyon parameters. This is a new observation.

V. DISCUSSION

Presence of fractional statistics has been predicted for quantum Spin Hall liquid and anyon excitations have been observed experimentally in some semiconductors. It seems natural to study the effects from anyon dynamics point of view. In the present work, following Horvathy et. al[18], we have shown that the anomalous velocity of Bloch electron in a semiclassical analysis, (resulting in Spin Hall effect), emerges naturally when equations of motion for anyons are studied in external electromagnetic field. A Non-Commutative phase space structure governs the anyon dynamics. In both the frameworks Berry

curvature plays a key role. We have computed the explicit forms of the Berry potentials adopting the method suggested in [10] and also in [18]. The analysis indicates that there is a possible connection between our results and physical properties of Ga-As alloy such as anomalous Hall conductivity. There is a natural and consistent relation between the parameters of our model (*e.g.* spin, mass and gyromagnetic ratio) with those of the bulk system (*e.g.* magnetic length, filling fraction) [20]. Our interpretation of anyon parameters can suggest connection with condensed matter systems qualitatively but more more work is needed for quantitative estimates. Possibility of measuring the anyon spin and gyromagnetic ratio from experimental observations is a new prediction of our scheme.

Acknowledgements: We thank Peter Horvathy and Diptiman Sen for correspondence.

[1] S.A.Wolf et.al., Science 294 (2001)1488.

[2] S.Murakami, N.Nagaosa and S.-C.Zhang, Science 301 (2003)1348; J.Sinova et.al., Phys.Rev.Lett. 92 (2004)126603. For a review, see S.Murakami, cond-mat/0504353.

[3] Y.K.Kato et.al., Science 306 (2004) ; J.Wunderlich et.al., Phys.Rev.Lett. 94 (2005) 047204.

[4] B.A.Bernevig and S.-C.Zhang, cond-mat/0504147.

[5] See for example F.Wilczek, *Fractional statistics and anyon superconductivity*, World Scientific, Singapore, 1990.

[6] J.Ihm, Phys.Rev.Lett. 67 (1991)251; F.E.Camino, W.Zhou and V.J.Goldman, Phys.Rev.Lett. 95 (2005)246802; Eun-Ah Kim, Phys.Rev.Lett. 97, 216404 (2006); PRL 95, 176402 (2005) and cond-mat/0604325; Eun-Ah Kim, M.J.Lawler, S.Vishveshwara and E.Fradkin, Phys.Rev. B 74, 155324 (2006); Phys.Rev.Lett. 95, 176402 (2005).

[7] R.Karplus and J.M.Luttinger, Phys.Rev. 95 1154(1954).

[8] T.Jungwirth, Q.Niu and A.H.Macdonald, Phys.Rev.Lett. 88 207208(2002); M.Onoda and N.Nagaosa, Phys.Rev.Lett. 90 (2003)206601; Y. Taguchi, Y. Oohara, H. Yoshizawa, N. Nagaosa, Y. Tokura, Science 291, 2573 (2001); J. Sinova, T. Jungwirth, J. Kucera, A. H. MacDonald, Phys. Rev. B 67, 235203 (2003); Z.Fang, N. Nagaosa, Kei S. Takahashi, A. Asamitsu, R. Mathieu, T. Ogasawara, H. Yamada, M. Kawasaki, Y. Tokura, K. Terakura, Science 302, 92 (2003)

[9] M.C.Chang and Q.Niu, Phys.Rev.Lett. 75, 1348(1995), G.Sundaram and Q.Niu, Phys.Rev.B B59(1999)14915; A.Bohm et.al., *The Geometric Phase in Quantum Systems*, Springer, 2003.

[10] For a review, see K.Yu.Bliokh and Yu.P.Bliokh, Ann.Phys.(N.Y.) 319 (2005) 13.

[11] M.V.Berry, Proc.Roy.Soc. London A392 (1984)45; A.Shapere and F.Wilczek, *Geometric Phases in Physics*, World Scientific, 1989.

[12] Apart from the Anomalous and Spin Hall effects [2, 8], a number of phenomena can be studied through Berry curvature effect in a unified way, such as, the modified density of states of electrons in Anomalous Hall Effect (D.Xiao, J.Shi and Q.Niu, Phys.Rev.Lett. 95 (2005) 137204; C. Duval et.al., Phys.Rev.Lett. 96 (2006) 099701), existence of magnetic monopole (in momentum space) (M.Onoda and N.Nagaosa, J.Phys.Soc.Jpn. 71 19(2002); Z.Fang et al., Science 302 92(2003); F. D. M. Haldane, Phys. Rev. Lett. 93, 206602 (2004)), Hall Effect of light (V.S.Liberman and B.Ya.Zel'dovich, Phys.Rev. A46 (1992)5199; K.Yu. Bliokh and Yu.P.Bliokh, Phys.Lett. A333 (2004)181; M.Onoda, M.Murakami and N.Nagaosa, Phys.Rev.Lett. 93 (2004)083901; C.Duval, Z.Horvath, P.A.Horvathy, Phys.Rev. D74 (2006)021701(R); A.Berard and H.Mohrbach, Phys.Lett. A352 (2006)190.), photon-gravitation interactions (P.Gosselin, A.Berard and H.Mohrbach, hep-th/0603227.) etc..

[13] M.S.Plyushchay, Phys.Lett. B248 (1990)107; R.Jackiw and V.P.Nair, Phys.Rev D43 (1991)1933.

[14] C.Chou, V.P.Nair and A.P.Polychronakos, Phys.Lett. B304 (1993) 105.

[15] S.Ghosh, Phys.Lett. B338 (1994) 235; Erratum-ibid. B347 (1995) 468; Phys.Rev. D51 (1995) 5827-5829; Erratum-ibid. D52 (1995) 4762.

[16] C. Duval, P.A.Horvathy, Phys.Lett. B594 (2004) 402.

[17] N.Seiberg and E.Witten, JHEP 9909(1999)032. For reviews see for example M.R.Douglas and N.A.Nekrasov, Rev.Mod.Phys. 73(2001)977; R.J.Szabo, *Quantum Field Theory on Noncommutative Spaces*, hep-th/0109162.

[18] P.A.Horvathy et al., Mod. Phys. Lett. A20 (2005) 1177.

[19] P.Zhang and Q.Niu, cond-mat/0406436.

[20] Z.F.Ezawa, *Quantum Hall Effects Field Theoretical Approach and Related Topics*, (World Scientific) (2000).

[21] A.Berard and H.Mohrbach. Phys.Rev. D69 95 127701(2004).